About the Algorithm

The Math Pac finds the roots of polynomials using Laguerre's method, which is an iterative process. The Laguerre step at the iterate Z_k for the polynomial $P(Z)$ of degree N is

$$\frac{-NP(Z_k)}{P'(Z_k) \pm [(N-1)^2 (P'(Z_k))^2 - N(N-1) P(Z_k) P''(Z_k)]^{1/2}}$$

The sign in the denominator is chosen to give the Laguerre step of smaller magnitude. Polynomials or their quotients of degree < 3 are solved using the quadratic formula or linear factorization.

Laguerre's method is cubically convergent to isolated zeros and linearly convergent to zeros of multiplicity greater than one.

The PROOT function is global in the sense that the user is not required to supply either an initial guess or a stopping criterion; in other words, no prior knowledge of the location of the roots is assumed. The PROOT function always attempts to begin its search (iteration) at the origin of the complex plane. An annulus in the plane known to contain the smallest magnitude root of the current (original or quotient) polynomial is constructed about the origin (using five theoretical bounds) and the initial Laguerre step is rejected if it exceeds the upper limit of this annulus. In this case, a spiral search from the lower radius of the annulus in the direction of the rejected initial step is begun until a suitable initial iterate is found.

Once the iteration process has successfully started, circles around each iterate are constructed (using two theoretical bounds) that are known to bound the root closest to that iterate; the Laguerre step size is constantly tested against the radii of these circles and modification of the step is made when it is deemed to be too large or when the polynomial value does not decrease in the direction of the step. For this reason, the roots are normally found in order of increasing magnitude, thus minimizing the roundoff errors resulting from deflation.

Evaluation of the polynomial and its derivatives at a real iterate is exactly Horner's method. Evaluation at a complex iterate is a modification of Horner's method that saves approximately half of the multiplications. This modification takes advantage of the fact that the Horner recurrence is symmetric with respect to complex conjugation.

PROOT uses a sophisticated technique to determine when an approximation Z_k should be accepted as a root. As the polynomial is being evaluated at Z_k, a bound for the evaluation roundoff error is also being computed. If the polynomial value is less than this bound, Z_k is accepted as a root. Z_k can also be accepted as a root if the value of the polynomial is decreasing but the size of the Laguerre step has become negligible. Before an approximation Z_k is used in an evaluation, its imaginary part is set to zero if this part is small compared to the step size. This improves performance, since real-number evaluations are faster than complex evaluations. If the Laguerre step size has become negligible but the polynomial is not decreasing, then the message PROOT failure is reported and the computation stops. This is expected never to occur in practice.

As the polynomial is being evaluated, the coefficients of the quotient polynomial (by either a linear or quadratic factor corresponding to the Z_k) are also computed. When an approximation Z_k is accepted as a root, this quotient polynomial becomes the polynomial whose roots are sought, and the process begins again.

Multiple Zeros

No polynomial rootfinder, including PROOT, can consistently locate zeros of high multiplicity with arbitrary accuracy. The general rule-of-thumb for PROOT is that for multiple or nearly-multiple zeros, resolution of the root is approximately $12/K$ significant digits, where K is the multiplicity of the root.

Accuracy

PROOT's criterion for accuracy is that the coefficients of the polynomial reconstructed from the calculated roots should closely resemble the original coefficients.

We will illustrate PROOT's performance with isolated zeros using the 100th degree polynomial

$$P(Z) = \sum_{k=0}^{100} Z^k$$

Of the 200 real and imaginary components of the calculated roots, about half were found to 12 digit accuracy. Of the rest, the error did not exceed a few counts in the 12th digit.

The polynomial $(Z + 1)^{20}$ with all 20 roots equal to -1 was solved by PROOT to yield the following roots.

(−.997874038627,0)
(−.934656570635,0)
(−.947080146258,−.160105886062)
(−.947080146258,.160105886062)
(−.678701343788,−6.24034855342E−2)
(−.678701343788,6.24034855342E−2)
(−.815082852233,−.270565874916)
(−.815082852233,.270565874916)
(−.725960092383,−.178602450179)
(−.725960092383,.178602450179)
(−.934932478844,−.326980158732)
(−.934932478844,.326980158732)
(−1.06905713438,−.337946194292)
(−1.06905713438,.337946194292)
(−1.19977533452,−.295162714497)

(−1.19977533452,.295162714497)
(−1.30383056467,−.200016185042)
(−1.30383056467,.200016185042)
(−1.3593147483,7.00833934259E−2)
(−1.3593147483,−7.00833934259E−2)

The roots appear inherently inaccurate due to the high multiplicity of −1 as a root. Between 0 and 1 correct digits were expected, even though the first zero found was better than this. However, the reconstructed coefficients are very close and are shown below (rounded to 12 digits).

Original Coefficients	Reconstructed Coefficients
1	1
20	20
190	190.000000001
1140	1140
4845	4845.00000003
15504	15504
38760	38760.0000003
77520	77520.0000007
125970	125970.000001
167960	167960.000002
184756	184756.000002
167960	167960.000003
125970	125970.000002
77520	77520.0000015
38760	38760.0000009
15504	15504.0000004
4845	4845.00000011
1140	1140.00000004
190	190.000000042
20	20.0000000344
1	1.00000001018